Effect of noise on a particle moving in a periodic potential

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It is shown that for systems with a periodic potential, the flux is very sensitive to the strength of additive or/and multiplicative noise. Multiplicative noise becomes important when its strength is of the order of the barrier height, and it provides a means of additional control of the flux (voltage-current characteristics for a Josephson junction). In addition to a numerical analysis, the cases of weak and strong additive noise have also been considered analytically.

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For a particle in a periodical potential, described by the equation

$$\frac{dx}{dt}=a-b\sin x, \tag{1}$$

overdamped Brownian motion is a generic form suitable for describing many different physical phenomena, such as the Josephson junction [1], charge density waves [2], motion of fluxons in superconductors [3,4], and the ring-laser gyroscope [5]. In these various cases, x is the phase across the junction, position of the charge density wave, coordinate of fluxons, phase-angle difference between the clockwise and the counterclockwise running wave in a ring-laser gyroscope, while a is the bias current, rotation rate and potentials, respectively. Other applications of Eq. (1) include, among others, the phase looking in electric circuits [6], chemical reactions [7], oscillations in the visual cortex [8], penetration of biological channels by ions [6], and motion of defects in convective fluids [9].

The solution of Eq. (1) can be easily found. The most significant property of this solution is the threshold behavior of the flux, $dx/dt \equiv \dot{x}$,

$$\langle \dot{x}\rangle \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T \dot{x}\,dt=\begin{cases}0 & \text{for } a<b\\ \sqrt{a^2-b^2} & \text{for } a>b.\end{cases} \tag{2}$$

The content of this equation can be easily understood in terms of a pendulum. When the external torque a is small, the pendulum can perform only small oscillations around its equilibrium point, while for sufficiently large a, the pendulum is able to execute complete rotations.

So far we have considered only deterministic quantities. However, all physical parameters are subject to random perturbations that, roughly speaking, may have internal or external origin. The former (additive noise) will influence the parameter a in Eq. (1), whereas the latter (multiplicative noise) is responsible for fluctuations in b,

$$a=a_0+\sqrt{2D_1}\xi(t);\qquad b=b_0+\sqrt{2D_2}\eta(t). \tag{3}$$

Here, we assume for simplicity that $\xi(t)$ and $\eta(t)$ are the Gaussian white noise, $\langle\xi(t)\xi(t_1)\rangle=\langle\eta(t)\eta(t_1)\rangle=\delta(t-t_1)$. The influence of an additive noise $\xi(t)$ alone [$\eta(t)=0$] on the solution (2) has been demonstrated for the Josephson junction [1]. It turns out that the threshold effect (2) is blurred and $\langle\dot{x}\rangle\neq 0$ appears for all $a_0\neq 0$. Much less effort has been expended in studying the influence of only multiplicative noise $\eta(t)$ and in combination with additive noise. This is the aim of this note.

The Stratonovich interpretation of the Fokker-Planck equation for the probability distribution function $P(x,t)$ corresponding to Eqs. (1)–(3) has the following form [10]:

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}[a_0-(b_0+D_2\cos x)\sin x]P+\frac{\partial^2}{\partial x^2}[(D_1+D_2\sin^2 x)]P\equiv-\frac{\partial W}{\partial x}, \tag{4}$$

where W is the flux proportional to $\langle\dot{x}\rangle$, namely, $\langle\dot{x}\rangle=2\pi W$.

For the stationary case, $\partial P/\partial t=0$, one finds [11]

$$\frac{dP}{dx}+\Gamma(x)P=W\Omega^2(x), \tag{5}$$

where

$$\Gamma(x)=\frac{a_0-b_0\sin x-D_2\sin x\cos x}{(D_1+D_2\sin^2 x)};$$

$$\Omega(x)=(D_1+D_2\sin^2 x)^{-1/2}. \tag{6}$$

The solution of the first-order differential equation (5) contains one constant which, together with the second constant W, is found from the normalization condition, $\int_{-\pi}^{\pi}P(x)dx=1$, and the periodicity condition, $P(-\pi)=\pi)$. Calculations yield

$$\langle\dot{x}\rangle=2\pi\left[1-\exp\left(-\frac{2\pi a_0}{\sqrt{D_1(D_1+D_2)}}\right)\right]\left[\int_0^{2\pi}\Omega(x)F(x,0)\times\left(\int_x^{x+2\pi}\Omega(y)F(0,y)dy\right)dx\right]^{-1}, \tag{7}$$

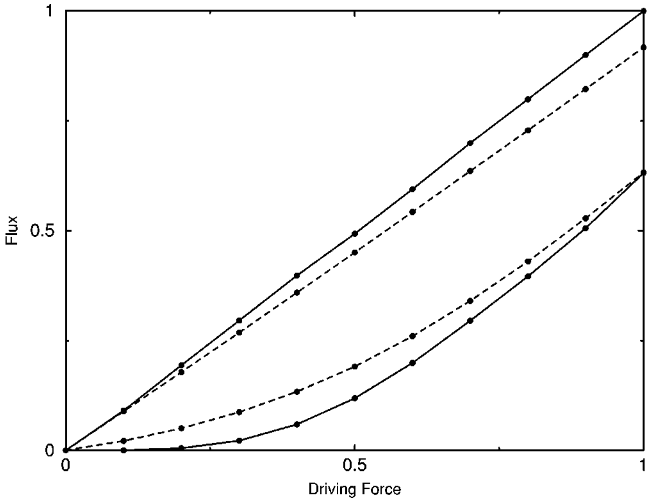

FIG. 1. Flux $\langle \dot{x} \rangle$ as a function of the driving force a_0 for $b_0 = 1$. Solid and dotted lines describe a single multiplicative noise and a single additive noise, respectively. The upper and lower curves are related to noises of strengths 2 and 0.1, respectively.

where

$$F(k,l) = \exp\left[-\int_k^l T(z)dz \right]; \quad T(z) = \frac{a_0 - b_0 \sin z}{D_1 + D_2 \sin^2 z}. \tag{8}$$

We initially performed numerical calculations of Eq. (7) in order to compare the importance of additive and multiplicative noise. In Fig. 1, we present the flux as a function of a driving force a_0, for $b_0 = 1$, in the presence of only one of noises equal to 0.5 and 2.0. From this figure one can see that for a small value of noise ($D = 0.5$), additive noise leads to a higher flux than multiplicative noise, whereas for larger noise ($D = 2$), the opposite result occurs. The transient regime takes place for intermediate value of noise. As it is shown in Fig. 2, for the noise $D = 1$ (of order of b_0), additive noise produces higher flux for small a_0 and smaller flux for larger a_0.

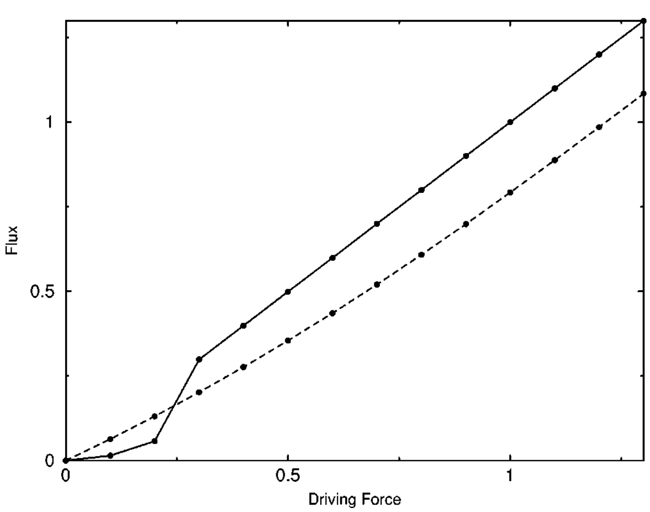

FIG. 2. The same as in Fig. 1 in the presence of a single noise of strength 1.

It is convenient for the following analysis to consider separately two limiting cases of weak ($D_1 \to 0$) and strong ($D_1 \to \infty$) additive noise, combining analytical and numerical approaches.

Let us start with the case of weak noise, $D_1 \to 0$ and $D_2 \to 0$, where both cases $D_2 > D_1$ and $D_2 < D_1$ are possible. One can use the method of steepest descent to calculate the integrals in Eq. (7), which gives

$$\langle \dot{x} \rangle = \left[1 - \exp\left(-\frac{2\pi a_0}{\sqrt{D_1(D_1+D_2)}} \right) \right] \times \frac{\sqrt{|T(z_{\max})\dot{T}(z_{\min})|}}{\Omega(z_{\max})\Omega(z_{\min})} \exp \int_{z_{\max}}^{z_{\min}} T(z)dz, \tag{9}$$

where $z_{\min}$ and $z_{\max}$ are two neighboring zeros of $T(z)$ with $T(z_{\max}) > 0$, $T(z_{\min}) < 0$.

It is easily found from Eq. (8) that $\sin(z_{\max,\min}) = a_0/b_0$, $\cos z_{\max} = \omega/b_0$, $\cos z_{\min} = -\omega/b_0$, and, for $b_0 > a_0$, Eq. (9) reduces to

$$\langle \dot{x} \rangle = \sqrt{b_0^2 - a_0^2} \left[1 - \exp\left(-\frac{2\pi a_0}{\sqrt{D_1(D_1+D_2)}} \right) \right] \times \exp \int_{z_{\max}}^{z_{\min}} T(z)dz. \tag{10}$$

Calculating the integral in Eq. (10) presents no problem, but instead of writing down this cumbersome expression, we present the results for the two limiting cases of large (small) multiplicative noise compared with additive noise, $D_2 \lessgtr D_1$.

For $D_2 < D_1$, i.e., for the weak additive and no multiplicative noise, one obtains the well-known result [12,1]

$$\langle \dot{x} \rangle_{D_2 < D_1} = \sqrt{b_0^2 - a_0^2} \exp\left(\frac{\pi a_0}{D_1} \right) \exp\left[-\frac{2\sqrt{b_0^2 - a_0^2}}{D_1} - \frac{2a_0}{D_1} \sin^{-1} \frac{a_0}{b_0} \right], \tag{11}$$

while for weak noise, with $D_2 > D_1$,

$$\langle \dot{x} \rangle_{D_2 > D_1} = \sqrt{b_0^2 - a_0^2} \exp\left(\frac{-\pi a_0}{\sqrt{D_1 D_2}} \right) \left(\frac{b_0 - \sqrt{b_0^2 - a_0^2}}{b_0 + \sqrt{b_0^2 - a_0^2}} \right)^{b_0/D_2}. \tag{12}$$

Comparing Eqs. (11) and (12) one concludes that additional multiplicative noise is able essentially to increase the flux in a system subject to only weak additive noise. These analytical results are supported by numerical analysis of Eq. (7), as shown in Fig. 3, for $D_1 = 0.1$ and different D_2, which shows the strong influence of multiplicative noise on the flux for small driving force.

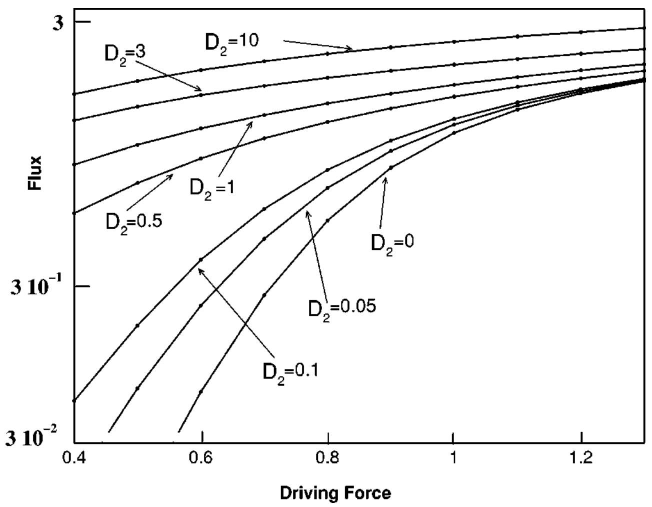

FIG. 3. Flux $\langle\dot{x}\rangle$ as a function of the driving force a_0 for $b_0=1$ and $D_1=0.1$ for different values of D_2.

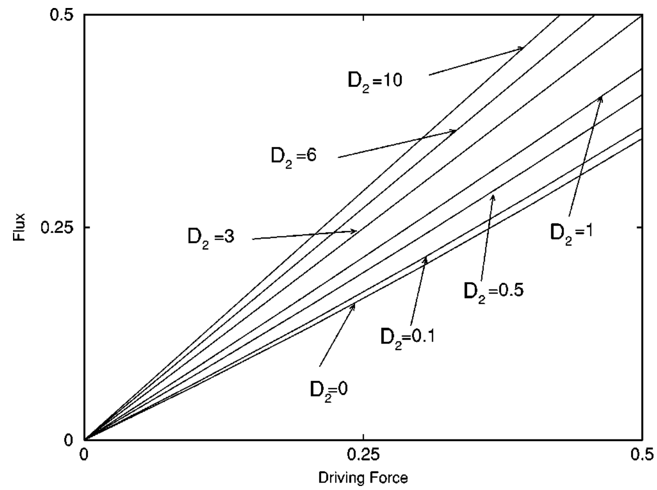

FIG. 5. Flux $\langle\dot{x}\rangle$ as a function of the driving force a_0 for $b_0=1$ and $D_1=1$ for different values of D_2.

Turning now to the opposite limiting case of strong additive noise, $D_1\to\infty$, one can substantially simplify Eq. (7), reducing it to the following form:

$$\langle\dot{x}\rangle_{D_1\to\infty}=\frac{a_0\pi^2}{D_1}\left(1+\frac{D_2}{D_1}\right)^{-1/2}\left[\int_0^{\pi}\Omega(z)dz\right]^{-2}. \quad (13)$$

In Fig. 4, we show the dimensionless flux $(\langle\dot{x}\rangle_{D_1\to\infty})/a_0$ as a function of D_2/D_1 for large additive noise D_1. This graph starts from $(\langle\dot{x}\rangle_{D_1\to\infty})/a_0=1$ for $D_2=0$ (large additive noise suppresses the sin term in Eq. (1), yielding Ohm's law for the Josephson junction [12]), increasing markedly with an increase in the strength of multiplicative noise.

Figure 5 shows the results of numerical analysis of Eq. (7) for comparable values of all parameters involved (b_0, D_1, and D_2), which again demonstrates an increase of the flux due to the multiplicative noise.

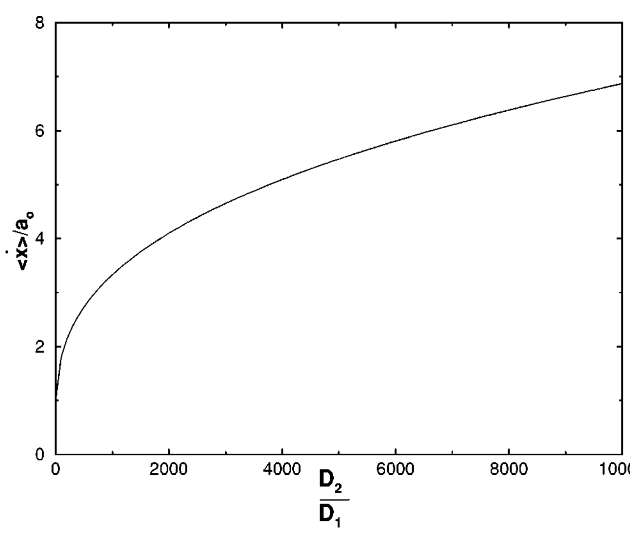

FIG. 4. Dimensionless flux $\langle\dot{x}\rangle/a_0$ as a function of the ratio of noise strengths D_2/D_1 for strong additive noise D_1 ($b_0=1$).

One concludes, therefore, that in the presence of one source of noise, the flux $\langle\dot{x}\rangle$ is larger for additive noise if the strength of noise is small, while for strong noise, multiplicative noise is more effective (Fig. 1). The transient regime between these two cases occurs for intermediate noise strength of order of b_0 (the critical current for Josephson junction), where additive noise is more effective for small driving forces and less effective than multiplicative noise for large driving forces (Fig. 2). In fact, for small noise strength (say, $D=0.1$) and small a_0, multiplicative noise produces flux larger by many order of magnitude than the flux caused by additive noise. It is not surprising that multiplicative noise becomes important when D is of order of the potential barrier height b_0.

If both sources of noise are present, then the flux is essentially increased in the presence of strong multiplicative noise for weak (Fig. 3), strong (Fig. 4) and intermediate (Fig. 5) strength of additive noise, especially for small bias force a_0. The latter result has a simple intuitive explanation. Indeed, the horizontal periodic potential ($a_0=0$) with a strong fluctuations in the width of this potential, has no preferential direction, and, therefore, no flux, $\langle\dot{x}\rangle=0$ (We leave aside the ratchet effect that requires spatially anisotropic periodic potential or/and nonequilibrium fluctuations.) It is enough to have a small slope of a periodic potential, $a_0\neq 0$, for the occurrence of the flux $\langle\dot{x}\rangle$.

The importance of multiplicative noise for the stationary states has long been known [13,14]. The influence of both additive and multiplicative noises on the escape time from a double-well potential was studied in [15,16]. The analysis of the stationary probability distribution function for a periodic potential and dichotomous multiplicative noise has been performed by Park *et al.* [17]. In 1997, we studied [11] the influence of both additive and multiplicative noises on the voltage-current characteristics of Josephson junctions. The similar effect for the different problem of an output-input relation for the motion in a double-well potential has been studied intensively by two groups of researchers, who called this effect ''noise-induced hypersensitivity'' [18] and ''amplification of weak signals via on-off intermittency'' [19].

We are looking for experimental verification of the comparative influence of two sources of noise, and the essential sensitivity of a flux to strong multiplicative noise for small bias in some of the systems [1–9] described by Eq. (1). It seems plausible that the described effect may have practical application for optimizing the flux in such systems.

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